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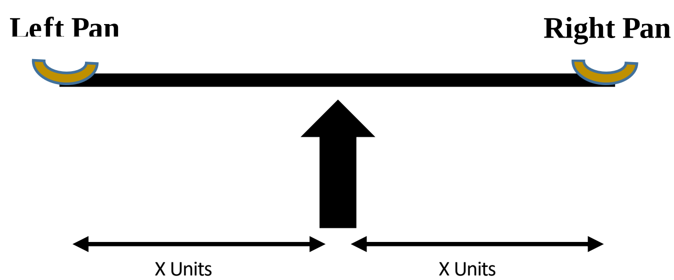
Numeral System and the Physical Balance

A numeral system is a mathematical representation of numbers based on a radix. The radix represents the number of unique symbols/digits that can be used in the mathematical representation. The radix could be any number such as 2 in the case of binary system, 3 in the case of ternary system or 10 in the case of decimal system. To represent a number based on a particular radix, the unique symbols/digits of the corresponding radix are used at one or more positions. And each position has a place-value which are in powers of the radix.

For example, the mathematical representation "100" (read as one-zero-zero) stands for number "four" in the binary system. In binary system, the place-value of the positions are in the powers of 2 (which is the radix). The place-value of the right-most position is 2^0 which is equal to 1, the one next to it has a place-value of 2^1 which is equal to 2, and so on. So it is clear that $100 = 2^2 * 1 + 2^1 * 0 + 2^0 * 0 = 4 + 0 + 0 = 4$.

Decimal system is the most commonly used numeral system all over the world. However, other numeral systems are also in use, in very specific areas. Ex: The binary system used in computers, and the sexagesimal system (Radix = 60) used in clocks. Interestingly, these other numeral system also form the mathematical basis behind many puzzles and math tricks. The aim of this article is to explore one such category of puzzles – the Weight Puzzles.

Let me take one simple puzzle. A shop keeper has a two pan physical balance. The weights can be kept only on the left pan, while the objects to be weighed can be kept only on the right pan. He visits a hardware shop to buy the weights. The hardware shop has weights of 1KG, 2KG, 3KG and so on till 15KG. What is the **minimum** number of weights that he needs to buy so that he would be able to weigh objects all the way from 1KG to 15KG?



Before I go ahead to reveal the answer, let us make one quick observation. The "weights" can be kept only on the left pan. So any given "weight" that the shopkeeper has, can be either

(a) Present on the left pan or
(b) Not present on the left pan.

Thus it can only be in one of these two unique states at any point of time. Representing each state by a special symbol/digit, we can easily see that the "radix" here is 2. This brings us to the binary system where the place-value of the binary digits (bits) are in the powers of two. If we have weights corresponding to the place-value of the bits in the binary system, we can weigh objects from 1KG till all the way up to the sum of all the weights. So if the shopkeeper buys these four weights - **1KG, 2KG, 4KG** and **8KG**, he can weigh objects all the way from 1KG to (1 + 2 + 4 + 8) = 15KG. This is the answer for the puzzle.

We can quickly check that to weigh an object of **9 KG**, the shopkeeper needs to keep **1KG** and **8KG** weights on the left pan. No wonder **9** = **1001** in the binary system.

Let me modify the puzzle slightly. The shopkeeper is allowed to keep the weights on any of the pans, and the object to be weighed on any one of the pans. In other words, the weights can be kept on the left pan or the right pan or both, and the objects to be weighed can be kept on either the left pan or the right pan. What is the **minimum** number of weights that he needs to buy so that he would be able to weigh objects all the way from 1KG to 40KG?

Applying the same reasoning, we can see that any given "weight" can be either

(a) Present on the left pan or
(b) Present on the right pan or
(c) Not present on either of the pans.

Note: The same given weight cannot be present on both the pans at the same time (obviously).

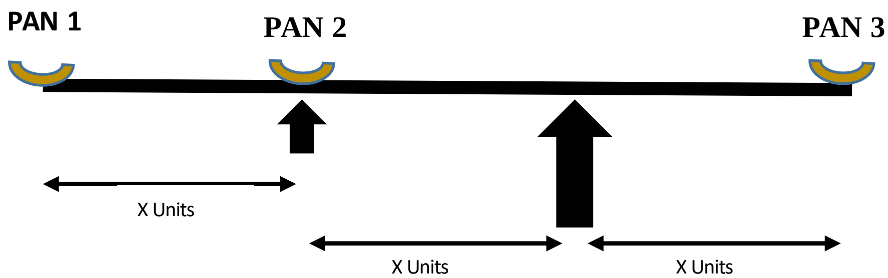
Thus it can only be in one of these three unique states at any point of time. Representing each state by a special symbol/digit, we can easily see that the "radix" here is 3. This brings us to the ternary system where the place-value of the ternary digits (trits) are in the powers of three. If we have weights corresponding to the place value of the trits in the ternary system, we can weigh objects from 1KG till all the way up to the sum of all the weights. So if the shopkeeper buys these four weights -**1KG, 3KG, 9KG** and **27KG**, he can weigh objects all the way from 1KG to (1 + 3 + 9 + 27) = 40KG. This is the answer for the puzzle.

Let us see how we can weigh an object of **15KG**. In the ternary system, 15 is represented as 120. To find out the placement of the weights on the pans, we need to follow these rules for each ternary digit starting from the rightmost trit (lowermost trit).

- **Rule 1**: If the trit is 0, do not do anything.
- **Rule 2**: If the trit is 1, then keep the weight corresponding to the place value of the trit on the left pan.
- **Rule 3**: If the trit is 2, then keep the weight corresponding to the place value of the trit on the right pan, and add 1 to the trit value which is on the immediate left (i.e., the next higher value trit).

Using the above technique, we can see that the shopkeeper would have to keep **27KG** on the left pan, but **9KG** and **3KG** on the right pan. The object to be weighed has to be then kept on the right pan. Once the pans are perfectly balanced, the object on the right pan would weigh 27 – (9 + 3) = 15 KG.

Wasn’t that cool? Let us extend the idea further. Consider a 3-Pan Balance as shown below.



At any given point, a “weight” can be either

(a) Present on the Pan-1 or
(b) Present on the Pan-2 or
(c) Present on the Pan-3 or
(d) Not present on any of the pans.

Thus it can only be in one of these four unique states at any point of time. So can you prove that just two weights - **1KG and 4KG** are sufficient to weigh objects all the way from **1KG** to **10KG** using this 3-Pan Balance? Do you see any additional parameter to be considered here? Can you construct an n-Pan Physical Balance and create your own weight puzzle based on that?

REFERENCE:

“Mathematical Approach To Puzzle Solving” by Kiran Ananthpur Bacche. [ISBN-13: 978-1945688201]